

GEOMETRY OF OBLIQUE PROJECTIONS ^{*†}

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Abstract

Let \mathcal{A} be a unital C^* -algebra. Denote by P the space of selfadjoint projections of \mathcal{A} . We study the relationship between P and the spaces of projections P_a determined by the different involutions $\#_a$ induced by positive invertible elements $a \in \mathcal{A}$. The maps $\varphi_p : P \rightarrow P_a$ sending p to the unique $q \in P_a$ with the same range as p and $\Omega_a : P_a \rightarrow P$ sending q to the unitary part of the polar decomposition of the symmetry $2q - 1$ are shown to be diffeomorphisms. We characterize the pairs of idempotents $q, r \in \mathcal{A}$ with $\|q - r\| < 1$ such that there exists a positive element $a \in \mathcal{A}$ verifying that $q, r \in P_a$. In this case q and r can be joined by an unique short geodesic along the space of idempotents Q of \mathcal{A} .

1 Introduction.

Let \mathcal{H} be a Hilbert space with scalar product \langle, \rangle . For every bounded positive invertible operator $a : \mathcal{H} \rightarrow \mathcal{H}$ consider the scalar product \langle, \rangle_a given by

$$\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

It is clear that \langle, \rangle_a induces a norm equivalent to the norm induced by \langle, \rangle . With respect to the scalar product \langle, \rangle_a , the adjoint of a bounded linear operator $x : \mathcal{H} \rightarrow \mathcal{H}$ is

$$x^{\#_a} = a^{-1}x^*a.$$

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Thus, x is a -selfadjoint if and only if

$$ax = x^*a.$$

Given a closed subspace S of \mathcal{H} , denote by $p = P_S$ the orthogonal projection from \mathcal{H} onto S and, for any positive operator a , denote by $\varphi_p(a)$ the unique a -selfadjoint projection with range S . In a recent paper, Z. Pasternak-Winiarski [20] proves the analyticity of the map $a \mapsto \varphi_p(a)$ and calculates its Taylor expansion. This study is relevant for understanding reproducing kernels of Hilbert spaces of holomorphic L^2 sections of complex vector bundles and the way they change when the measures and hermitian structures are deformed (see [21], [22]). This type of deformations appears in a natural way when studying quantization of systems where the phase space is a Kähler manifold (Odziejewicz [18], [19]).

In this paper we pose Pasternak-Winiarski's problem in the C^* -algebra setting and use the knowledge of the differential geometry of idempotents, projections and positive invertible elements in order to get more general results in a shorter way.

More precisely, let \mathcal{A} be a unital C^* -algebra, $G = G(\mathcal{A})$ the group of invertible elements of \mathcal{A} , $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$ the unitary group of \mathcal{A} , $G^+ = \{a \in G : a^* = a, a \geq 0\}$ the space of positive invertible elements of \mathcal{A} ,

$$Q = Q(\mathcal{A}) = \{q \in \mathcal{A} : q^2 = q\} \quad \text{and} \quad P = P(\mathcal{A}) = \{p \in Q : p = p^*\},$$

the spaces of idempotents and projections of \mathcal{A} . The nonselfadjoint elements of Q will be called oblique projections. It is well known that Q is a closed analytic submanifold of \mathcal{A} , P is a closed real analytic submanifold of Q and G^+ is an open submanifold of

$$\mathcal{S} = \mathcal{S}(\mathcal{A}) = \{b \in \mathcal{A} : b^* = b\},$$

which is a closed real subspace of \mathcal{A} (see [24], [7] or [9] for details).

We define a fibration

$$\varphi : P \times G^+ \rightarrow Q$$

which coincides, when $\mathcal{A} = L(\mathcal{H})$, with the map $(p, a) \mapsto \varphi_p(a)$, the unique a -selfadjoint projection with the same range as p . This allows us to study the analyticity of Pasternak-Winiarski's map in both variables p, a . The

rich geometry of Q , P and G^+ give an amount of information which may be useful in the problems that motivated [20].

Along this note we use the fact that every $p \in Q$ induces a representation α_p of elements of \mathcal{A} by 2×2 matrices given by

$$\alpha_p(a) = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}.$$

Under this homomorphism p can be identified with

$$\begin{pmatrix} 1_{p\mathcal{A}p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

and all idempotents q with the same range of p have the form

$$q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}.$$

for some $x \in p\mathcal{A}(1-p)$. This trivial remark shortens many proofs in a drastic way and the analyticity of some maps (for example $\varphi : P \times G^+ \rightarrow Q$) follows immediately.

The contents of the paper are the following. Section 2 contains some preliminary material including the matrix representations mentioned above and the description of the adjoint operation induced by each positive invertible (element or operator) a .

In section 3 we study the map $\varphi_p = \varphi(p, \cdot) : G^+ \rightarrow Q$, which is Pasternak-Winiarski's map when \mathcal{A} is $L(\mathcal{H})$ and p is the orthogonal projection P_S onto a closed subspace $S \subseteq \mathcal{H}$. For $a \in G^+$, let $P_a = P_a(\mathcal{A})$ denote the set of all $\#_a$ -selfadjoint projections. This is a subset of Q and section 4 starts a study of the relationship between $P = P_1$ and P_a and the way they are located in Q . In particular we show that $\varphi_\alpha = \varphi(\cdot, a) : P \rightarrow P_a$ is a diffeomorphism and compute its tangent map. Another interesting map is the following: for $q \in P_a$, $\varepsilon = 2q - 1$ is a reflection, i.e. $\varepsilon^2 = 1$, which admits in \mathcal{A} a polar decomposition $\varepsilon = \lambda\rho$, with $\lambda \in G^+$ and ρ a unitary element of \mathcal{A} . It is easy to see that $\rho = \rho^* = \rho^{-1}$ so that $p = \frac{1}{2}(\rho + 1) \in P$. In section 5 we prove that the map $\Omega_a : P_a \rightarrow P$ given by $\Omega_a(q) = p$ is a diffeomorphism and study the movement of P given by the composition $\Omega_a \circ \varphi_\alpha : P \rightarrow P$. We also characterize the orbit of p by these movements, i.e.

$$\mathcal{O}_p := \{r \in P : \Omega_a \circ \varphi_a(p) = r \text{ for some } a \in G^+\}.$$

In recent years several papers have appeared which study length of curves in P and Q (see [25], [3], [23], [7], [2] for example). It is known that P and the fibres of $\Omega : Q \rightarrow P$ are geodesically complete and their geodesics are short curves (for convenient Finsler metrics -see [7]). For a fixed $p \in P$, let us call horizontal (resp. vertical) those directions around p which produce geodesics along P (resp. along the fiber $\Omega^{-1}(p)$). In section 6 we show that there exist short geodesics in many other directions (not only the horizontal and the vertical ones).

This paper, which started from a close examination of Pasternak-Winiarski's work, is part of the program of understanding the structure of the space of idempotent operators. For a sample of the vast bibliography on the subject the reader is referred to the papers by Afriat [1], Kovarik [15], Zemánek [29], Porta-Recht [24], Gerisch [11], Corach [6] and the references therein. Applications of oblique projections to complex, harmonic and functional analysis and statistics can be found in the papers by Kerzman and Stein [13], [14], Ptak [27], Coifman and Murray [5] and Mizel and Rao [17], among others.

2 Preliminary results.

Let \mathcal{H} be a Hilbert space, $\mathcal{A} \subset L(\mathcal{H})$ a unital C^* -algebra, $G = G(\mathcal{A})$ the group of invertible elements and $\mathcal{U}_{\mathcal{A}}$ the unitary group of \mathcal{A} ,

If S is a closed subspace of \mathcal{H} and q is a bounded linear projection onto S , then

$$p = qq^*(1 - (q - q^*)^2)^{-1} \quad (1)$$

is the unique selfadjoint projection onto S . Note that, by this formula, $p \in \mathcal{A}$ when $q \in \mathcal{A}$. Several different formulas are known for p (see [11], p. 294); perhaps the simplest one is the so-called Kerzman-Stein formula

$$p = q(1 + q - q^*)^{-1} \quad (2)$$

(see [13], [14] or [5]). However, for the present purposes, (1) is more convenient. We denote by

$$Q = Q(\mathcal{A}) = \{q \in \mathcal{A} : q^2 = q\} \text{ and } P = P(\mathcal{A}) = \{p \in \mathcal{A} : p = p^* = p^2\} \quad (3)$$

the spaces of idempotents and projections of \mathcal{A} . Given a fixed closed subspace S of \mathcal{H} , we denote by

$$Q_S = Q_S(\mathcal{A}) = \{q \in Q(\mathcal{A}) : q(\mathcal{H}) = S\} \quad (4)$$

the space of idempotents of \mathcal{A} with range S . Note that, by (1), Q_S is not empty if and only if the projection $p = p_S$ onto S belongs to \mathcal{A} . We shall make this assumption.

It is easy to see that two idempotents $q, r \in Q$ have the same range if and only if $qr = r$ and $rq = q$. Therefore the space Q_S of (4) can be characterized as

$$Q_S = Q_p = \{q \in Q : qp = p, pq = q\}.$$

In what follows, we shall adopt this notation Q_p , emphasizing the role of p , rather than S . This will enable us to simplify many computations. Moreover this operator algebraic viewpoint allows one to get the results below independently of the representation of \mathcal{A} .

Recall some facts about matrix representations. Every $p \in Q$ induces a representation α_p of elements of \mathcal{A} by 2×2 matrices given by

$$\alpha_p(a) = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}. \quad (5)$$

If $p \in P$ the representation preserves the involution $*$. For simplicity we shall identify a with $\alpha_p(a)$ and \mathcal{A} with its image by α_p . Observe that, with this convention,

$$p = \begin{pmatrix} 1_{p\mathcal{A}p} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

Moreover, $q \in Q_S = Q_p$ if and only if there exists $x \in p\mathcal{A}(1-p)$ such that

$$q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}. \quad (7)$$

Indeed, let $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q_p$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p = qp = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

then $a = 1$ and $c = 0$. On the other hand

$$q = pq = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix},$$

then $d = 0$ and b can be anything. We summarize this information in the following:

Proposition 2.1 *The space Q_p can be identified with $p\mathcal{A}(1 - p)$ by means of the affine map*

$$Q_p \rightarrow p\mathcal{A}(1 - p), \quad q \mapsto q - p \quad (8)$$

Proof. Clearly, the affine map defined in (8) is injective. By (7) it is well defined and onto. ■

In the Hilbert space \mathcal{H} , every scalar product which is equivalent to the original \langle, \rangle is determined by a unique positive invertible operator $a \in L(\mathcal{H})$ by means of

$$\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}. \quad (9)$$

For this scalar product the adjoint $x^{\#_a}$ of $x \in L(\mathcal{H})$ is easily seen to be

$$x^{\#_a} = a^{-1}x^*a \quad (10)$$

where $*$ denotes the adjoint operation for the original scalar product. Operators which are selfadjoint for some $\#_a$ have been considered by Lax [16] and Dieudonné [10]. A geometrical study of families of C^* -involutions has been done by Porta and Recht [26].

Denote by $G^+ = G^+(\mathcal{A})$ the set of all positive invertible elements of \mathcal{A} . Every $a \in G^+$ induces as in (10) a continuous involution $\#_a$ on \mathcal{A} by means of $x^{\#_a} = a^{-1}x^*a$, for $x \in \mathcal{A}$. \mathcal{A} is a C^* -algebra with the involution $\#_a$ and the corresponding norm $\|x\|_a = \|a^{1/2}xa^{1/2}\|$ for $x \in \mathcal{A}$. The mapping $x \mapsto a^{-1/2}xa^{1/2}$ is an isometric isomorphism of $(\mathcal{A}, \|\cdot\|, *)$ onto $(\mathcal{A}, \|\cdot\|_a, \#_a)$. In this setting, \mathcal{A} can be also represented by the inclusion map in $L(\mathcal{H}, \langle, \rangle_a)$.

Note that the map $a \mapsto \langle, \rangle_a \mapsto \#_a$ is not one to one, since (10) says that if $a \in \mathbb{C}I$ then $\#_a = *$. If we regard this map in G^+ with values in

the set of involutions of \mathcal{A} , then two elements $a, b \in G^+$ with $a = bz$ for z in the center of \mathcal{A} ,

$$\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} : zc = cz, \quad \text{for all } c \in \mathcal{A}\}, \quad (11)$$

produce the same involution $\#_a$.

2.2 Recall the properties of the conditional expectation induced by a fixed projection $p \in P$. Note that the set \mathcal{A}_p of elements of \mathcal{A} which commute with p is the C^* -subalgebra of \mathcal{A} of diagonal matrices in terms of the representation (5). We denote by $E_p : \mathcal{A} \rightarrow \mathcal{A}_p \subset \mathcal{A}$ the conditional expectation defined by compressing to the diagonal:

$$E_p(a) = pap + (1-p)a(1-p) = \begin{pmatrix} pap & 0 \\ 0 & (1-p)a(1-p) \end{pmatrix}, \quad a \in \mathcal{A}.$$

This expectation has the following well known properties ([28], Chapter 2): for all $a \in \mathcal{A}$,

1. $E_p(bac) = bE_p(a)c$ for all $b, c \in \mathcal{A}_p$.
2. $E_p(a^*) = E_p(a)^*$.
3. If $b \leq a$ then $E_p(b) \leq E_p(a)$. In particular $E_p(G^+) \subset G^+$.
4. $\|E_p(a)\| \leq \|a\|$.
5. If $0 \leq a$, then $2E_p(a) \geq a$.

3 Idempotents with the same range.

The main purpose of this section is to describe, for a fixed $p \in P$, the map which sends each $a \in G^+$ into the unique $q \in Q_p$ which is $\#_a$ -selfadjoint. This problem was posed and solved by [20] when $\mathcal{A} = L(\mathcal{H})$. Here we use 2×2 matrix arguments to give very short proofs of the results of [20]. Moreover we generalize these results and apply them to understand some aspects of the geometry of the space Q .

Let us fix the notations: For each $a \in G^+$ denote by

$$\mathcal{S}_a = \mathcal{S}_a(\mathcal{A}) = \{b \in \mathcal{A} : b^{\#_a} = b\}, \quad (12)$$

the set of $\#_a$ -selfadjoint elements of \mathcal{A} .

Definition 3.1 Let \mathcal{A} be a C^* -algebra and $p \in P$ a fixed projection of \mathcal{A} . We consider the map

$$\varphi_p : G^+ \rightarrow Q_p \quad \text{given by } \varphi_p(a) = \text{the unique } q \in Q_p \cap \mathcal{S}_a, \quad a \in G^+.$$

Note that existence and uniqueness of such q follow from (1) applied to the C^* -algebra \mathcal{A} with the star $\#_a$.

Proposition 3.2 Let \mathcal{A} be a C^* -algebra and $p \in P$. Then, for all $a \in G^+(\mathcal{A})$,

$$\varphi_p(a) = pE_p(a)^{-1}a, \quad (13)$$

where E_p is the conditional expectation defined in (2.2). In particular,

$$\|\varphi_p(a)\| \leq 2 \|a\| \|a^{-1}\|.$$

Proof. Suppose that, in matrix form, we have

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} \quad \text{and then} \quad E_p(a) = \begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix}.$$

Since $\varphi_p(a) \in Q_p$, by (7) there exists $x \in p\mathcal{A}(1-p)$ such that $\varphi_p(a) = p+x$. On the other hand, by (10), $p+x \in \mathcal{S}_a$ if and only if $a^{-1}(p+x)^*a = p+x$, i.e. $(p+x^*)a = a(p+x)$. In matrix form,

$$(p+x^*)a = \begin{pmatrix} 1 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ x^*a_1 & x^*a_2 \end{pmatrix} \quad \text{and}$$

$$a(p+x) = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_1x \\ a_2^* & a_2^*x \end{pmatrix}.$$

Then $(p+x^*)a = a(p+x)$ if and only if $a_2 = a_1x$. Note that $a \in G^+(\mathcal{A})$ implies that $a_1 \in G^+(p\mathcal{A}p)$. Then

$$\varphi_p(a) = \begin{pmatrix} 1 & a_1^{-1}a_2 \\ 0 & 0 \end{pmatrix}, \quad (14)$$

and now formula (13) can be proved by easy computations. Finally, since $2E_p(a) \geq a$, we deduce that $E_p(a)^{-1} \leq 2a^{-1}$ and the inequality $\|\varphi_p(a)\| \leq 2\|a\|\|a^{-1}\|$ follows easily. ■

Remark 3.3 There is a way to describe φ_p in terms of (2) with the star $\#_a$. In this sense we obtain, for $p \in P$ and $a \in G^+$,

$$\varphi_p(a) = p(1 + p - a^{-1}pa)^{-1} = p(a + ap - pa)^{-1}a.$$

Clearly $a + ap - pa = E_p(a) + 2a_2^*$ and one obtains (13), since $p(a + ap - pa)^{-1} = pE_p(a)^{-1}$. However it seems difficult to obtain bounds for $\|\varphi_p(a)\|$ by using this approach.

3.4 Consider the space G^+ as an open set of $\mathcal{S} = \mathcal{S}(\mathcal{A}) = \mathcal{S}_1(\mathcal{A})$, the closed real subspace of selfadjoint elements of \mathcal{A} . Then the map $\varphi_p : G^+ \rightarrow \mathcal{A}$ is real analytic. Indeed, if $h \in \mathcal{S}$ and $\|h\| < 1$, then

$$\varphi_p(1 + h) = p(1 + E_p(h))^{-1}(1 + h) = p \sum_{n=0}^{\infty} (-1)^n E_p(h)^n (1 + h), \quad (15)$$

and this formula is clearly real analytic near 1. More computations starting from (15) give the more explicit formula

$$\varphi_p(1 + h) = p + \sum_{n=1}^{\infty} (-1)^{n-1} (ph)^n (1 - p), \quad (16)$$

again for all $h \in \mathcal{S}$ with $\|h\| < 1$. These computations are very similar to those appearing in the proof of Theorem 5.1 of [20]. We include them for the sake of completeness. By (15),

$$\begin{aligned} \varphi_p(1 + h) &= \sum_{n=0}^{\infty} (-1)^n (php)^n (p + ph) \\ &= \sum_{n=0}^{\infty} (-1)^n (php)^n + \sum_{n=0}^{\infty} (-1)^n (ph)^{n+1} \\ &= p + \sum_{n=1}^{\infty} (-1)^n (ph)^n p + \sum_{n=1}^{\infty} (-1)^{n-1} (ph)^n \\ &= p + \sum_{n=1}^{\infty} (-1)^{n-1} (ph)^n (1 - p). \end{aligned}$$

As a consequence (see also Theorem 3.1 of [20]) the tangent map $(T\varphi_p)_1 : \mathcal{S} \rightarrow \mathcal{A}$ is given by

$$(T\varphi_p)_1(X) = pX(1-p) \quad \text{for } X \in \mathcal{S}. \quad (17)$$

Actually, by (2.8) Q_p is an affine manifold parallel to the closed subspace $p\mathcal{A}(1-p)$ which can be also regarded as its “tangent” space. In this sense $(T\varphi_p)_1$ is just the natural compression of \mathcal{S} onto $p\mathcal{A}(1-p)$.

Note that formulas (15) and (16) do not depend on the selected star in \mathcal{A} . Using this fact, formula (16) can be generalized to a power series around each $a \in G^+$ by using (16) with the star $\#_a$ at $q = \varphi_p(a)$. Indeed, note that for every $b \in G^+$, $<, >_b = (<, >_a)_{a^{-1}b}$ is induced from $<, >_a$ by $a^{-1}b$, which is a -positive. If $h \in \mathcal{S}$ and $\|h\| < \|a^{-1}\|^{-1}$, then $a + h \in G^+$, $\|a^{-1}h\|_a = \|a^{-1/2}ha^{-1/2}\| \leq \|h\|\|a^{-1}\| < 1$ and

$$\varphi_p(a + h) = \varphi_q(1 + a^{-1}h) = q + \sum_{n=1}^{\infty} (-1)^{n-1} (qa^{-1}h)^n (1-q), \quad (18)$$

showing the real analyticity of φ_p in G^+ and also giving the way to compute the tangent map $(T\varphi_p)_a$ at every $a \in G^+$.

Formulas (17), (18) and their consequence, the real analyticity of φ_p for $\mathcal{A} = L(\mathcal{H})$, are the main results of [20]. Here we generalize these results to an arbitrary C^* -algebra \mathcal{A} . In the following section, we shall explore some of their interesting geometrical interpretations and applications.

4 Differential geometry of Q .

The space Q of all idempotents of a C^* -algebra (or, more generally, of a Banach algebra) has a rich topological and geometrical structure, studied for example in [17], [29], [11], [24], [7] and [8].

We recall some facts on the structure of Q as a closed submanifold of \mathcal{A} . The reader is referred to [7] and [8] for details. The tangent space of Q at q is naturally identified to

$$\begin{aligned} \{X \in \mathcal{A} : qX + Xq = X\} &= \{X \in \mathcal{A} : qXq = (1-q)X(1-q) = 0\} \\ &= q\mathcal{A}(1-q) \oplus (1-q)\mathcal{A}q. \end{aligned} \quad (19)$$

In terms of the matrix representation induced by q ,

$$T(Q)_q = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in \mathcal{A} \right\} \quad (20)$$

The set P is a real submanifold of Q . The tangent space $(TP)_p$ at $p \in P$ is

$$\{X \in \mathcal{A} : pX + Xp = X, \ X^* = X\},$$

which in terms of the matrix representation induced by p is

$$T(P)_p = \left\{ \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in \mathcal{A} \right\} = T(Q)_p \cap \mathcal{S}. \quad (21)$$

The space Q (resp. P) is a discrete union of homogeneous spaces of G (resp. \mathcal{U}_A) by means of the natural action

$$G \times Q \rightarrow Q \quad \text{given by} \quad (g, q) \mapsto gqg^{-1} \quad (22)$$

(resp. $\mathcal{U}_A \times \mathcal{P} \rightarrow \mathcal{P}$, $(u, p) \mapsto upu^*$).

There is a natural connection on Q (resp. P) which induces in the tangent bundle TQ (resp. TP) a linear connection. The geodesics of this connection, i.e. the curves γ such that the covariant derivative of $\dot{\gamma}$ vanishes, can be computed. For $X \in (TQ)_p$ (resp. $(TP)_p$), the unique geodesic γ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$ is given by

$$\gamma(t) = e^{tX'} p e^{-tX'},$$

where $X' = [X, p] = Xp - pX$. Thus, the exponential map $\exp_p : T(Q)_p \rightarrow Q$ is given by

$$\exp_p(X) = e^{X'} p e^{-X'}, \quad \text{for} \quad X \in T(Q)_p. \quad (23)$$

Proposition 4.1 *The inverse of the affine bijective map*

$$\Gamma : Q_p \rightarrow p\mathcal{A}(1 - p), \quad \Gamma(q) = q - p.$$

of (8) is the restriction of the exponential map at p to the closed subspace $p\mathcal{A}(1 - p) \subset T(Q)_p$. That is, for $x \in p\mathcal{A}(1 - p)$, $\exp_p(x) = p + x \in Q_p$.

Proof. Let $x \in p\mathcal{A}(1-p)$. Then

$$\begin{aligned}
\exp_p(x) &= \exp_p \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\
&= \exp \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix} p \exp \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \quad \text{by (23)} \\
&= \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} p \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = p + x. \quad \blacksquare
\end{aligned}$$

Remark 4.2 The map φ_p of 3.1 can be also described using Proposition 4.1. In fact, consider the real analytic map

$$u_p : G^+ \rightarrow G(\mathcal{A}) \quad \text{given by} \quad u_p(a) = \exp(-pE_p(a)^{-1}a(1-p)), \quad a \in G^+.$$

Then, by 4.1, $\varphi_p(a) = u_p(a)pu_p(a)^{-1}$. This is an explicit formula of an invertible element which conjugates p with $\varphi_p(a)$. This can be a useful tool for lifting curves of idempotents to curves of invertible elements of \mathcal{A} .

Now we consider the map φ_p by letting p vary in P :

$$\varphi : P \times G^+ \rightarrow Q \quad \text{given by} \quad \varphi(p, a) = \varphi_p(a) = pE_p(a)^{-1}a, \quad (24)$$

for $p \in P$, $a \in G^+$. Consider also the map $\phi : Q \rightarrow P$ given by (1):

$$\phi(q) = qq^*(1 - (q - q^*)^2)^{-1}, \quad \text{for} \quad q \in Q. \quad (25)$$

This map ϕ assigns to any $q \in Q$ the unique $p \in P$ with the same range as q .

Proposition 4.3 *The map $\varphi : P \times G^+ \rightarrow Q$ is a C^∞ fibration. For $q \in Q$, let $p = \phi(q)$ and $x = q - p \in p\mathcal{A}(1-p)$. Then the fibre of q is*

$$\varphi^{-1}(q) = \left\{ (p, \begin{pmatrix} a_1 & a_1x \\ x^*a_1 & a_3 \end{pmatrix}) : 0 < a_1 \text{ and } x^*a_1x < a_3 \right\}, \quad (26)$$

where the inequalities of the right side are considered in $p\mathcal{A}p$ and $(1-p)\mathcal{A}(1-p)$, respectively. Moreover, the fibration φ splits by means of the C^∞ global cross section

$$s : Q \rightarrow P \times G^+ , \quad \text{given by} \quad s(q) = (\phi(q), |2q - 1|), \quad (27)$$

for $q \in Q$, where $|z| = (z^*z)^{1/2}$.

Proof. Let us first verify (26). Fix $q \in Q$. The only possible first coordinate of every pair in $\varphi^{-1}(q)$ must be $p = \phi(q)$, since it is the unique projection in P with the same range as q .

Given $a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix} \in G^+$, we know by (14) that $\varphi(p, a) = q$ if and only if $a_2 = a_1(q - p) = a_1x$. Then $a = \begin{pmatrix} a_1 & a_1x \\ x^*a_1 & a_3 \end{pmatrix}$. The inequalities $x^*a_1x < a_3$ in $(1-p)\mathcal{A}(1-p)$ and $a_1 > 0$ in $p\mathcal{A}p$ are easily seen to be equivalent to the fact that $\begin{pmatrix} a_1 & a_1x \\ x^*a_1 & a_3 \end{pmatrix} \in G^+$. This shows (26).

Denote by $\varepsilon = 2q - 1$. It is clear that $\varepsilon^2 = 1$, i.e. ε is a symmetry. Consider its polar decomposition $\varepsilon = \rho\lambda$, where $\lambda = |\varepsilon| \in G^+$ and ρ is a unitary element of \mathcal{A} . From the uniqueness of the polar decomposition it follows that $\rho = \rho^* = \rho^{-1}$, i.e. ρ is a unitary selfadjoint symmetry. Then, since $q = \frac{\varepsilon+1}{2}$,

$$\lambda^{-1}q^*\lambda = \lambda^{-1}\left(\frac{\varepsilon^*+1}{2}\right)\lambda = \lambda^{-1}\left(\frac{\lambda\rho+1}{2}\right)\lambda = \frac{\rho\lambda+1}{2} = \frac{\varepsilon+1}{2} = q.$$

Therefore $q \in \mathcal{S}_\lambda(\mathcal{A})$ and $\varphi(p, \lambda) = \varphi(s(q)) = q$ proving that s is a cross section of φ ■

4.4 The space P is the selfadjoint part of the space Q . But each $a \in G^+$ induces the star $\#_a$ and therefore another submanifold of Q of a -selfadjoint idempotents. Let $a \in G^+$ and denote the a -selfadjoint part of Q by

$$P_a = P_a(\mathcal{A}) = \{ q \in Q : q^{\#_a} = q \}. \quad (28)$$

We are going to relate the manifolds P and P_a . There is an obvious way of mapping P onto P_a , namely $p \mapsto a^{-1/2}pa^{1/2}$. Its tangent map is the

restriction of the isometric isomorphism $X \mapsto a^{-1/2} X a^{1/2}$ from \mathcal{S} onto \mathcal{S}_a mentioned in section 2. We shall study some less obvious maps between P and P_a .

For a fixed $a \in G^+$, consider the map

$$\varphi_a : P \rightarrow P_a \quad \text{given by} \quad \varphi_a(p) = \varphi(p, a), \quad p \in P. \quad (29)$$

Then φ_a is a diffeomorphism between the submanifolds P and P_a of Q and φ_a^{-1} is just the map ϕ of (25) restricted to P_a . The problem which naturally arises is the study of the tangent map of φ_a in order to compare different P_a , $a \in G^+$.

The tangent space $(TP_a)_q$ for $q \in P_a$ can be described as in (21),

$$(TP_a)_q = \left\{ Y = \begin{pmatrix} 0 & y \\ y^{\#_a} & 0 \end{pmatrix} \in \mathcal{A} \right\} = T(Q)_q \cap \mathcal{S}_a, \quad (30)$$

where the matricial representations are in terms of q . Therefore any $Y \in (TP_a)_q$ is characterized by its 1,2 entry $y = qY$ by the formula

$$Y = y + y^{\#_a} = qY + Yq. \quad (31)$$

Proposition 4.5 *Let $p \in P$, $a \in G^+$ and $X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (TP)_p$. Denote by $q = \varphi_a(p) \in P_a$. Then, in terms of p ,*

$$q (T\varphi_a)_p(X) = \begin{pmatrix} 0 & a_1^{-1}x(a_3 - a_2^*a_1^{-1}a_2) \\ 0 & 0 \end{pmatrix} = y.$$

Therefore $(T\varphi_a)_p(X) = y + y^{\#_a}$ and $\|(T\varphi_a)_p(X)\|_a = \|y\|_a$.

Proof. We have the formula of Proposition 3.2,

$$\varphi_a(p) = \varphi_p(a) = pE_p(a)^{-1}a = p (pap + (1-p)a(1-p))^{-1}a.$$

By the standard method of taking a smooth curve γ in P such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$, one gets

$$(T\varphi_a)_p(X) = \left[X - pE_p(a)^{-1}(Xap + paX - Xa(1-p) - (1-p)aX) \right] E_p(a)^{-1}a.$$

Since p and $E_p(a)$ commute, $pE_p(a)^{-1}(1-p)aX = 0$. In matrix form in terms of p , by direct computation it follows that

$$\begin{aligned}
(T\varphi_a)_p(X) &= \\
&= \left[\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} - \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} xa_2^* + a_2x^* & a_1x - xa_3 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & a_1^{-1}a_2 \\ a_3^{-1}a_2^* & 1 \end{pmatrix} \\
&= \begin{pmatrix} -a_1^{-1}xa_2^* - a_1^{-1}a_2x^* & a_1^{-1}xa_3 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} 1 & a_1^{-1}a_2 \\ a_3^{-1}a_2^* & 1 \end{pmatrix} \\
&= \begin{pmatrix} -a_1^{-1}a_2x^* & a_1^{-1}(xa_3 - xa_2^*a_1^{-1}a_2 - a_2x^*a_1^{-1}a_2) \\ x^* & x^*a_1^{-1}a_2 \end{pmatrix}.
\end{aligned}$$

Multiplying by $q = \varphi(p, a)$, by (14), one obtains

$$\begin{aligned}
q (T\varphi_a)_p(X) &= \\
&= \begin{pmatrix} 1 & a_1^{-1}a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -a_1^{-1}a_2x^* & a_1^{-1}(xa_3 - xa_2^*a_1^{-1}a_2 - a_2x^*a_1^{-1}a_2) \\ x^* & x^*a_1^{-1}a_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_1^{-1}x(a_3 - a_2^*a_1^{-1}a_2) \\ 0 & 0 \end{pmatrix} = y,
\end{aligned}$$

as desired. The fact that $\|y\|_a = \|Y\|_a$ is clear by regarding them as elements of $(\mathcal{A}, \#_a)$ and using (30).

5 The polar decomposition.

In this section it is convenient to identify Q with the set of symmetries (or reflections) $\{\varepsilon \in \mathcal{A} : \varepsilon^2 = 1\}$ and P with the set of selfadjoint symmetries $\{\rho \in \mathcal{A} : \rho = \rho^* = \rho^{-1}\}$ by means of the affine map $x \mapsto 2x - 1$.

Recall that every invertible element c of a unital C^* -algebra admits polar decompositions $c = \rho_1\lambda_1 = \lambda_2\rho_2$, with $\lambda_1, \lambda_2 \in G^+$ and $\rho_1, \rho_2 \in \mathcal{U}_{\mathcal{A}}$. Moreover,

$$\lambda_1 = |c|, \lambda_2 = |c^*| \quad \text{and} \quad \rho_1 = \rho_2 = |c^*|^{-1}c = c|c|^{-1}.$$

In particular, if ε is a symmetry, its polar decompositions are $\varepsilon = |\varepsilon^*| \rho = \rho |\varepsilon|$ and

$$\rho = \rho^* = \rho^{-1} \in P. \quad (32)$$

This remark defines the retraction

$$\Omega : Q \rightarrow P, \quad \text{by } \Omega(\varepsilon) = \rho. \quad (33)$$

The map Ω has been studied from a differential geometric viewpoint in [7]. If $\varepsilon \in Q$, it is easy to show that $|\varepsilon^*| = |\varepsilon|^{-1}$ and $|\varepsilon^*|^{1/2} \rho = \rho |\varepsilon^*|^{-1/2}$ (see [7]). This section is devoted to study, for each $a \in G^+$, the restriction

$$\Omega_a = \Omega|_{P_a} : P_a \rightarrow P. \quad (34)$$

Observe that, with the identification mentioned above, $P_a = Q \cap \mathcal{S}_a = Q \cap \mathcal{U}_a$, where $\mathcal{U}_a = \{u \in G : u^{-1} = u^{\#_a}\}$ is the group of a -unitary elements of \mathcal{A} .

Proposition 5.1 *For every $a \in G^+$ the map $\Omega_a : P_a \rightarrow P$ of (34) is a diffeomorphism.*

Proof. By the remarks above, for every $\varepsilon \in Q$

$$\Omega_a(\varepsilon) = \rho = |\varepsilon| \varepsilon \quad (35)$$

which is clearly a C^∞ map.

Set $b = a^{1/2}$ and consider, for a fixed $\rho \in P$, the polar decomposition of $b\rho b$ given by $b\rho b = w|b\rho b|$, with $w \in \mathcal{U}_a$. Since $b\rho b$ is invertible and selfadjoint by (32), it is easy to prove (see [9]) that

$$w = w^* = w^{-1} \in P, \quad w b\rho b = b\rho b w \quad \text{and} \quad w b\rho b = |b\rho b| \in G^+.$$

Let $\varepsilon = b^{-1}wb$. It is clear by its construction that $\varepsilon \in P_a$. Also $\varepsilon\rho = \lambda > 0$, since

$$b\varepsilon\rho b = w b\rho b = |b\rho b| \in G^+.$$

Therefore the polar decomposition of ε must be $\varepsilon = \lambda\rho$. So $\lambda = |\varepsilon^*|$ and $\Omega_a(\varepsilon) = \rho$. Therefore

$$\Omega_a^{-1}(\rho) = a^{-1/2} \left(a^{1/2} \rho a^{1/2} |a^{1/2} \rho a^{1/2}|^{-1} \right) a^{1/2}, \quad (36)$$

which is also a C^∞ map, showing that Ω_a is a diffeomorphism. \blacksquare

Remark 5.2 The fibres of the retraction Ω over each $p \in P$ are in some sense, “orthogonal” to P . In order to explain this remark, consider the algebra $\mathcal{A} = M_n(\mathbb{C})$ of all $n \times n$ matrices with complex entries. Then $M_n(\mathbb{C})$ has a natural scalar product given by $\langle X, Y \rangle = \text{tr}(Y^*X)$. It is easy to prove that for every $p \in P$, $(TP)_p$ is orthogonal to $(T\Omega^{-1}(P))_p$. The same result holds in every C^* -algebra with a trace τ . Then the map $a \mapsto \Omega_a(\rho)^{-1}$ of (36) can be considered as the “normal” movement which produces a -selfadjoint projections for every $a \in G^+$.

On the other hand, the map φ_p of (14), which was studied also in [20], gives another way to get a -selfadjoint projections for every $a \in G^+$. In terms of the geometry of Q this way is, in the sense above, an oblique movement. A related movement is to take for each $a \in G^+$, an a -selfadjoint projection q' with $\ker q' = \ker p$.

Combining, for a fixed $a \in G^+$ the maps φ_a of (29) and Ω_a of (34), one obtains a C^∞ movement of the space P . The following proposition describes explicitly this movement.

Proposition 5.3 *Let $a \in G^+$. Then the map $\Omega_a \circ \varphi_a : P \rightarrow P$ is a diffeomorphism of P . For $p \in P$, let $\varphi_a(p) = q = p + x$ and $\varepsilon = 2q - 1$. In terms of p , $x = a_1^{-1}a_2$ if $a = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix}$ and*

$$\begin{aligned} \Omega_a \circ \varphi_a(p) &= \begin{pmatrix} 1 + xx^* & 0 \\ 0 & 1 + x^*x \end{pmatrix}^{-1/2} \begin{pmatrix} 1 & x \\ x^* & -1 \end{pmatrix} \\ &= [qq^* + (1 - q)^*(1 - q)]^{-1/2} (q + q^* - 1). \end{aligned} \quad (37)$$

Proof. In matrix form, $\varepsilon = 2\varphi_a(p) - 1 = \begin{pmatrix} 1 & 2x \\ 0 & -1 \end{pmatrix}$ so that

$$\varepsilon^* \varepsilon = \begin{pmatrix} 1 & 0 \\ 2x^* & -1 \end{pmatrix} \begin{pmatrix} 1 & 2x \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 2x^* & 4x^*x + 1 \end{pmatrix} = |\varepsilon|^2.$$

On the other hand, by (27), $q \in P_{|\varepsilon|}(\mathcal{A})$. Therefore, by (14),

$|\varepsilon| = \begin{pmatrix} b & bx \\ x^*b & c \end{pmatrix}$ with b, c positive. Straightforward computations show that

$$b = (1 + xx^*)^{-1/2} \quad \text{and} \quad c^2 = 4x^*x + 1 - x^*(1 + xx^*)^{-1}x.$$

Since $x^*(1 + xx^*) = (1 + x^*x)x^*$,

$$\begin{aligned}
c^2 &= 4x^*x + 1 - (1 + x^*x)^{-1}x^*x \\
&= 4x^*x + (1 + x^*x)^{-1} \\
&= (1 + x^*x)^{-1}(4(x^*x)^2 + 4x^*x + 1) \\
&= (1 + x^*x)^{-1}(2x^*x + 1)^2.
\end{aligned}$$

Then $c = (1 + x^*x)^{-1/2}(2x^*x + 1)$ and

$$|\varepsilon| = \begin{pmatrix} 1 + xx^* & 0 \\ 0 & 1 + x^*x \end{pmatrix}^{-1/2} \begin{pmatrix} 1 & x \\ x^* & 2x^*x + 1 \end{pmatrix}. \quad (38)$$

Now the two formulas of (37) follow by easy matrix computations. \blacksquare

Remark 5.4 It is interesting to observe that the factor $q + q^* - 1$ of (37) has been characterized by Buckholtz [4] as the inverse of $P_{R(q)} - P_{\ker q}$.

A natural question about these movements is the following: for $p \in P$, how far can $\Omega_a \circ \varphi_a(p)$ be from p . In order to answer this question we consider the orbit

$$\mathcal{O}_p := \{r \in P : \Omega_a \circ \varphi_a(p) = r \text{ for some } a \in G^+\}. \quad (39)$$

The next result is a metric characterization of \mathcal{O}_p based on some results about the “unit disk” of the projective space of \mathcal{A} defined by p (see [2]).

Proposition 5.5 *Let $p \in P$. Then*

$$\mathcal{O}_p = \{r \in P : \|r - p\| < \frac{\sqrt{2}}{2}\}$$

Proof. Fix $a \in G^+$. Let $q = \varphi_a(p)$, $\varepsilon = 2q - 1$ and $r = \Omega_a \circ \varphi_a(p)$. By (27), r is also obtained if we replace a by $|\varepsilon|$, since $\varphi_a(p) = q = \varphi_{|\varepsilon|}(p)$ and $r = \Omega_a(q) = \Omega(q) = \Omega_{|\varepsilon|}(q)$. Note that $|\varepsilon|$ is positive and ρ -unitary, i.e. unitary for the signed inner product $<, >_\rho$ given by $\rho = |\varepsilon|\varepsilon = 2r - 1$. Indeed, by (32), $|\varepsilon|^{\#_\rho} = \rho^{-1}|\varepsilon|\rho = |\varepsilon|^{-1}$.

Since $\varepsilon = \rho|\varepsilon| = |\varepsilon|^{-1/2}\rho|\varepsilon|^{1/2}$, also $q = |\varepsilon|^{-1/2}r|\varepsilon|^{1/2}$. In [2] it is shown that the square root of a ρ -unitary is also ρ -unitary. Then $|\varepsilon|^{-1/2}$ is ρ -unitary. In [2] it is also shown that

$$\|r - P_{R(\lambda r \lambda^{-1})}\| < \frac{\sqrt{2}}{2}$$

for all positive ρ -unitary λ . Note that $p = P_{R(q)}$ and then it must be $\|p - r\| < \frac{\sqrt{2}}{2}$. In Proposition 6.13 of [2] it is shown that for all $r \in P$ such that $\|r - p\| < \frac{\sqrt{2}}{2}$, there exists a positive $(2r - 1)$ -unitary λ such that $p = P_{R(\lambda r \lambda^{-1})}$. In this case $r = \Omega_\lambda \circ \varphi_\lambda(p) \in \mathcal{O}_p$. ■

6 New short geodesics.

Lengths of geodesics in P have studied in [25], [3], [23] and [2]. It has been proved that if $p, r \in P$ and $\|p - r\| < 1$, then there exists a unique geodesic of P joining them which has minimal length. On the other hand, the fibres $\Omega^{-1}(P)$ are geodesically complete and the geodesic joining $q_1, q_2 \in \Omega^{-1}(P)$ is a shortest curve in Q [8]. This final section is devoted to show the existence of “short oblique geodesics”, i.e. geodesics which are not contained neither in P nor in the fibres.

More precisely, the idea of the present section is to use different stars $\#_a$ for $a \in G^+$ in order to find short curves between pairs of non selfadjoint idempotents of \mathcal{A} . Basically we want to characterize those pairs $q, r \in Q$ such that there exist $a \in G^+$ with $q, r \in P_a$. If q and r remain close in P_a , they can be joined by a short curve in the space P_a .

The first problem is that the positive a need not be unique. This can be fixed up in the following manner:

Lemma 6.1 *Suppose that $a \in G^+$ and $p, r \in P \cap P_a$. Then $\|p - r\| = \|p - r\|_a$ and, if $\|p - r\| < 1$, the short geodesics which join them in P and P_a are the same and have the same length.*

Proof. Note that $P \cap P_a$ is the space of projections commuting with a . Let $\mathcal{B} = \{a\}' \cap \mathcal{A}$, the relative commutant of a in \mathcal{A} . Since $a = a^*$, \mathcal{B} is a C^* -algebra. Moreover, $P \cap P_a = P(\mathcal{B})$. Now, since $\|p - r\| < 1$, p and q can

be joined by the unique short geodesic γ along $P(\mathcal{B})$ (see [25] or [2]) and γ is also a geodesic both for P and P_a . The length of γ is computed in the three algebras in terms of the *norm* of the corresponding tangent vector X . But since $X \in \mathcal{B}$, its norm is the same with the two scalar products involved.

■

We shall give a characterization of pairs of close idempotents $p, q \in Q$ such that $p, q \in P_a$ for some $a \in G^+$. The characterization will be done in terms of a tangent vector $X \in T(Q)_p$ such that $q = e^X p e^{-X}$. First we give a slight improvement of the way to obtain such X which appears in 2) of [25]:

Proposition 6.2 *Let $p \in P$ and $q \in Q$ with $\|p - q\| < 1$. Let $\varepsilon = 2q - 1$, $\rho = 2p - 1$,*

$$v_1 = \frac{\varepsilon\rho + 1}{2} = qp + (1 - q)(1 - p) \quad \text{and} \quad v_2 = \frac{\rho\varepsilon + 1}{2} = pq + (1 - p)(1 - q).$$

Then $\|v_1 - 1\| = \|v_2 - 1\| = \|p - q\| < 1$ and

$$X = (Id - E_p)(\log v_1) = \frac{1}{2} (\log v_1 - \log v_2) \quad (40)$$

verifies that $X \in T(Q)_p$ (i.e. $pXp = (1 - p)X(1 - p) = 0$) and $q = e^X p e^{-X}$.

Proof. Note that $\rho = \rho^* = \rho^{-1} \in P$. Then

$$\|v_1 - 1\| = \left\| \frac{\varepsilon\rho - 1}{2} \right\| = \frac{1}{2} \|(\varepsilon - \rho)\rho\| = \|q - p\| < 1,$$

and similarly for v_2 . Let $X_i = \log v_i$ for $i = 1, 2$. Since $v_1\rho = \rho v_2$, and each X_i is obtained as a power series in v_i , we obtain also that $X_1\rho = \rho X_2$. Then, if $X = \frac{1}{2}(X_1 - X_2)$, we have that $X\rho = -\rho X$, and then $X \in T(Q)_p$.

Note also that $v_1\rho = \varepsilon v_1$ and $\|v_i - 1\| < 1$ for $i = 1, 2$. So $v_1 p v_1^{-1} = q$. Easy calculations show that v_1 and v_2 commute. As before this implies that X_1 and X_2 commute. Then

$$w = v_1 v_2 = v_2 v_1 = e^{X_1 + X_2} = \left(\frac{\varepsilon + \rho}{2} \right)^2$$

commutes with v_1 , v_2 , ρ , ε , p and q . Denote by

$$w^{-\frac{1}{2}} = e^{-\frac{X_1 + X_2}{2}}.$$

Since $(X_1 + X_2)\rho = \rho(X_1 + X_2)$, $w^{-1/2}$ commutes with ρ . Note that

$$X = \frac{X_1 - X_2}{2} = X_1 - \frac{X_1 + X_2}{2}. \quad (41)$$

This implies that $e^X = e^{X_1}w^{-1/2} = v_1w^{-1/2}$ and therefore

$$e^X p e^{-X} = v_1 p v_1^{-1} = q.$$

Finally, since X has zeros in its diagonal and $\frac{X_1 + X_2}{2}$ is diagonal in terms of p , we deduce from (41) that $X = (Id - E_p)(X_1)$ and the proof is complete. \blacksquare

Proposition 6.3 *Let $p \in P$ and $q \in Q$ such that $\|p - q\| < 1$. Let $X = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in T(P)_p$ as in (40), such that $e^X p e^{-X} = q$. Then the following are equivalent:*

1. *There exists $a \in G^+$ such that $p, q \in P_a$.*
2. *There exists $a \in G^+$ such that $pa = ap$ and $X^{\#_a} = -X$.*
3. *There exist $b \in G^+(p\mathcal{A}p)$ and $c \in G^+((1-p)\mathcal{A}(1-p))$ such that*

$$y = -cx^*b.$$

Proof. Condition 2. can be written as

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad a^{-1}X^*a = -X.$$

In matrix form

$$\begin{aligned} a^{-1}X^*a &= \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & y^* \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_1^{-1}y^*a_2 \\ a_2^{-1}x^*a_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -x \\ -y & 0 \end{pmatrix}, \end{aligned}$$

which clearly is equivalent to condition 3.

Condition 1. holds if $X^{\#_a} = -X$, since in that case e^X is a -unitary and then $q \in P_a$. In order to prove the converse, we consider $\#_a$ instead of $*$ and so condition 1. means that $p, q \in P$. Then, with the notations of (6.2), we have that $v_2 = v_1^*$ and

$$X_2 = \log v_2 = \log v_1^* = X_1^* \Rightarrow X^* = \frac{X_2 - X_1}{2} = -X,$$

showing 2. ■

Remark 6.4

Let us call a direction (i.e. tangent vector) in Q “good” if it is the direction of a short geodesic. Proposition 6.3 provides a way to obtain good directions. Other good directions occur in the spaces $p\mathcal{A}(1-p)$ and $(1-p)\mathcal{A}p$, determined by the affine spaces of projections with the same range (Q_p) or the same kernel as p , where the straight lines can be considered as short geodesics.

Still another good directions can be found looking at pairs $p, q \in Q$ such that, for some $a \in G^+$, $\Omega^a(q) = p$, where Ω^a means the retraction of (33), considering in \mathcal{A} the star $\#_a$. These pairs can be characterized in a very similar way as Proposition 6.3. In fact, in condition 3 (with the same notations), $y = -bx^*c$ should be replaced by $y = bx^*c$. These directions are indeed good because it is known [8] that along the fibers of each Ω^a there are short geodesics that join any pair of elements (not only close pairs).

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